THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Advanced Calculus I
Tutorial 8
Date: 11 June, 2025
Review Midterm Problems

1. Point out YES or NO without any proof for each statement.
   (a) If a function f is real-valued function defined on $\mathbb{R}$ and is differentiable everywhere, then f' is continuous.
   (b) For any integrable $f(x)$ defined on $[-1,1]$, then $F(x) = \int_{-1}^{x} f(t)dt$ is differentiable on $[-1,1]$.
   (c) At most two subsets of the real numbers are both open and closed.
   (d) If a function f is real-valued function defined on $\mathbb{R}^2$ and all directional derivatives exist and equal, then f is differentiable.
   (e) There exists a function f with continuous second-order partial derivatives such that $f_x(x,y) = 5x^2 + y$ and $f_y(x,y) = 5x^2 - y$.

2. (a) For any $u, v, w \in \mathbb{R}^3$, prove the identity

$$u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u).$$

   b) Find the volume of a parallelepiped with one of its eight vertices at $A(0,0,0)$, and three adjacent vertices at $B(1,2,0), C(0,-3,2)$ and $D(3,-4,5)$.

3. (a) Let $\gamma(t) = (\sin t + \cos t, 2\sin^2 t)$, for $t \in [0, \pi/2]$, be a curve in $\mathbb{R}^2$. Find the length of the curve γ. (No need to evaluate the final value of the integral.)
   (b) If a differentiable path $r(t)$ lies on a sphere centred at the origin, show that $r(t) \cdot r'(t) = 0$.

4. (a) Find the distance from the point $(0,0,0)$ to the plane $7x + 4y + 6z = 6$.
   (b) Find the equation of the tangent plane of the surface

$$z = 2x^3 + 6xy^2 - 3x^2 + 3y^2 - 2x - 2y$$

   at $(1,1,4)$.

5. (a) By using the Triangle inequlaity, or otherwise, show that for any $a, b, c, d \in \mathbb{R}$

$$\sqrt{(a+b)^2 + (b+c)^2 + (c+d)^2 + (a+d)^2} \leq 2\sqrt{a^2 + b^2 + c^2 + d^2}.$$

   (b) Suppose $a = 1$. Find all possible values of b, c and d so that the inequality above becomes an equality, i.e.,

$$\sqrt{(a+b)^2 + (b+c)^2 + (c+d)^2 + (a+d)^2} = 2\sqrt{a^2 + b^2 + c^2 + d^2}.$$

6. Evaluate the following limits or show they do not exist.

(a)
$$\lim_{(x,y)\to(0,0)} \frac{e^{x+y}-1}{xy}.$$

(b)
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

(c)
$$\lim_{(x,y)\to(0,0)} \frac{x^{12}y^2}{(x^6+y^2)^3}.$$

7. Let $f(x,y) = e^{\cos(2x^2+y^2)+2x+y}$.
   (a) Find the linearization of f at $(0,0)$.
   (b) Approximate $f(-0.3, 0.3)$ using the linearizatinon in the first problem.

8. Let
$$f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$
   (a) Show that $f(x,y)$ is continuous on $\mathbb{R}^2$.
   (b) Prove that $D_u f(0,0)$ exist for all unit vectors $u \in \mathbb{R}^2$.
   (c) Is $f(x,y)$ differentiable at $(0,0)$?

1. Point out YES or NO without any proof for each statement.

   (a) If a function f is real-valued function defined on $\mathbb{R}$ and is differentiable everywhere, then f' is continuous.

   (b) For any integrable $f(x)$ defined on $[-1, 1]$, then $F(x) = \int_{-1}^{x} f(t)dt$ is differentiable on $[-1, 1]$.

   (c) At most two subsets of the real numbers are both open and closed.

   (d) If a function f is real-valued function defined on $\mathbb{R}^2$ and all directional derivatives exist and equal, then f is differentiable.

   (e) There exists a function f with continuous second-order partial derivatives such that $f_x(x, y) = 5x^2 + y$ and $f_y(x, y) = 5x^2 - y$.

a) No., consider $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

b) No.

c) Exactly two subsets of $\mathbb{R}$ are both open and closed, $\mathbb{R}$, $\emptyset$.

d) No. $f(x,y) = \begin{cases} \frac{x^3 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

For any unit vector u, $D_u f(0,0) = 0$. but f is not diff. at 0.

e) No. $f_{xy} \neq f_{yx}$.

2. (a) For any $u, v, w \in \mathbb{R}^3$, prove the identity

$$u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u).$$

b) Find the volume of a parallelepiped with one of its eight vertices at $A(0,0,0)$, and three adjacent vertices at $B(1,2,0), C(0,-3,2)$ and $D(3,-4,5)$.

a) $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)$, $w=(w_1,w_2,w_3)$

$$u\cdot(v\times w) = u\cdot\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= u\cdot\left(v_2w_3-v_3w_2,\ -(v_1w_3-v_3w_1),\ v_1w_2-v_2w_1\right)$$

$$= u_1(v_2w_3-v_3w_2)+u_2(v_3w_1-v_1w_3)+u_3(v_1w_2-v_2w_1)$$

$$= u_1v_2w_3+u_2v_3w_1+u_3v_1w_2-(u_1v_3w_2+u_2v_1w_3+u_3v_2w_1)$$

Similarly:

$$w\cdot(u\times v) = \dots = w_1u_2v_3+w_2u_3v_1+w_3u_1v_2-(w_1u_3v_2+w_2u_1v_3+w_3u_2v_1)$$

So $u\cdot(v\times w) = w\cdot(u\times v)$ and the other equality is similar.

b) $AB=(1,2,0)$
$AC=(0,-3,2)$
$AD=(3,-4,5)$.

$$V=|AB\cdot(AC\times AD)|$$

$$=\left|\begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 3 & -4 & 5 \end{vmatrix}\right|$$

$$=\dots=5$$

3. (a) Let $\gamma(t) = (\sin t + \cos t, 2\sin^2 t)$, for $t \in [0, \pi/2]$, be a curve in $\mathbb{R}^2$. Find the length of the curve γ. (No need to evaluate the final value of the integral.)

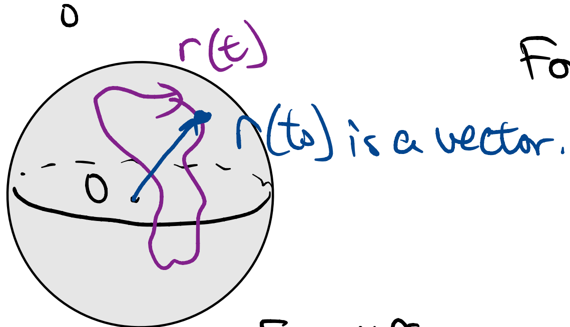
   (b) If a differentiable path $r(t)$ lies on a sphere centred at the origin, show that $r(t) \cdot r'(t) = 0$.

a) $L = \int_0^{\frac{\pi}{2}} |\gamma'(t)| dt$. $\quad \gamma'(t) = (\cos t - \sin t, 4\sin t\cos t)$.

$$\gamma'(t)\cdot\gamma'(t) = (\cos t - \sin t)^2 + (4\sin t\cos t)^2$$
$$= 1 - 2\cos t\sin t + 16\sin^2 t\cos^2 t.$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{1 - 2\cos t\sin t + 16\sin^2 t\cos^2 t}\, dt.$$

b)



For every t, $\|r(t)\|^2 = \text{Const}^2$.

So for every t,

$r(t)\cdot r(t) = \text{const}$.

So differentiating both sides in t, we get

$2r'(t)\cdot r(t) = 0 \Rightarrow r'(t)\cdot r(t) = 0$ for all t.

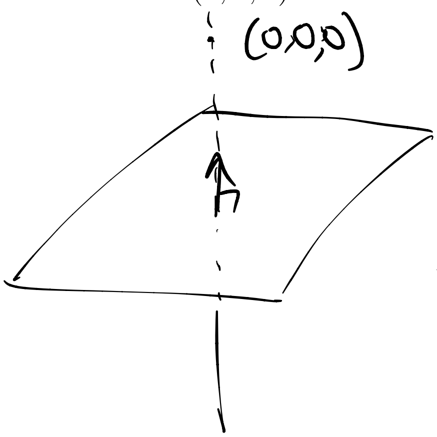
4. (a) Find the distance from the point $(0, 0, 0)$ to the plane $7x + 4y + 6z = 6$.

   (b) Find the equation of the tangent plane of the surface

$$z = 2x^3 + 6xy^2 - 3x^2 + 3y^2 - 2x - 2y$$

   at $(1, 1, 4)$.

a)



$l(t) = (0,0,0) + t(7,4,6) = (7t, 4t, 6t)$

Find t_0 s.t. $l(t_0)$ is on the plane.

$7(7t_0) + 4(4t_0) + 6(6t_0) = 6$

$101 t_0 = 6 \Rightarrow t_0 = \frac{6}{101}$.

$l(t_0) = \left(\frac{42}{101}, \frac{24}{101}, \frac{36}{101}\right)$.

$\|l(t_0)\| = \left\|\left(\frac{42}{101}, \frac{24}{101}, \frac{36}{101}\right)\right\| = \frac{6}{\sqrt{101}}$

b) Let $z = f(x,y)$ where $f(x,y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2 - 2x - 2y$

Then equation of tangent plane at $(x_0, y_0, f(x_0,y_0))$

is $z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

Check: $f(1,1) = 4$.

$f_x = 6x^2 + 6y^2 - 6x - 2 \quad f_x(1,1) = 4$

$f_y = 12xy + 6y - 2 \quad f_y(1,1) = 16$.

So eqn. of tangent plane is

$z = 4 + 4(x-1) + 16(y-1) = 4x + 16y - 16$.

5. (a) By using the Triangle inequlaity, or otherwise, show that for any $a, b, c, d \in \mathbb{R}$

$$\sqrt{(a+b)^2+(b+c)^2+(c+d)^2+(a+d)^2} \leq 2\sqrt{a^2+b^2+c^2+d^2}.$$

(b) Suppose $a = 1$. Find all possible values of b, c and d so that the inequality above becomes an equality, i.e.,

$$\sqrt{(a+b)^2+(b+c)^2+(c+d)^2+(a+d)^2} = 2\sqrt{a^2+b^2+c^2+d^2}.$$

a) $\text{LHS} = \|(a+b, b+c, c+d, a+d)\|$

$= \|(a,b,c,d) + (b,c,d,a)\|$

triangle ineq. $\leq \|(a,b,c,d)\| + \|(b,c,d,a)\|$

both vectors have the same length $= 2\|(a,b,c,d)\| = \text{RHS}$.

b) $a=1$. Recall equality case of Cauchy Schwarz inequality is when the two vectors are linearly dependent ($u = \lambda v$ for some $\lambda \geq 0$).

So we need,

$(b, c, d, 1) = \lambda(1, b, c, d)$

$b = \lambda$.

$c = \lambda b = \lambda^2$ $\Rightarrow 1 = \lambda^4 \Rightarrow \lambda = 1$.

$d = \lambda c = \lambda^3$ So $b = c = d = 1$ as well.

$1 = \lambda d = \lambda^4$

6. Evaluate the following limits or show they do not exist.

(a)
$$\lim_{(x,y)\to(0,0)} \frac{e^{x+y}-1}{xy}.$$

(b)
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

(c)
$$\lim_{(x,y)\to(0,0)} \frac{x^{12}y^2}{(x^6+y^2)^3}.$$

a) Consider the line $y=mx$.

$$\lim_{(x,y)\to(0,0)} \frac{e^{x+y}-1}{xy} = \lim_{x\to 0} \frac{e^{(1+m)x}-1}{mx^2}$$

$$\overset{\text{Taylor expand around 0}}{=} \lim_{x\to 0} \frac{(1+m)x}{mx^{2}}$$ diverges as $x\to 0$.

CORRECTION:

(Another way: use L'Hopital's Rule ~~twice~~ to get

$$\ldots = \lim_{x\to 0} \frac{e^{(1+m)x}(1+m)^2}{2m} = \frac{(1+m)^2}{2m}$$ ~~which depends on m, hence lim DNE.~~

NOT CORRECT to use L'Hopital's Rule twice since

$$\lim_{x\to 0} \frac{(e^{(1+m)x}-1)'}{(mx^2)'} = \lim_{x\to 0} \frac{(1+m)e^{(1+m)x}}{2mx}$$ is not an indeterminate form.

only need to use L'Hopital's Rule once since one can then see that $\lim_{x\to 0} \frac{(1+m)e^{(1+m)x}}{2mx}$ diverges.

b) Use polar coords: $x = r\cos\theta$, $y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{\cancel{r^2}\cos\theta\sin\theta}{\cancel{r^2}} = \lim_{r\to 0} \cos\theta\sin\theta = \cos\theta\sin\theta$$

depends on θ, so lim DNE.

e) $$\lim_{(x,y)\to(0,0)} \frac{x^{12}y^2}{(x^6+y^2)^3}$$

1) Consider $y = x^3$, then

$$\lim_{x\to 0} \frac{x^{18}}{(2x^6)^3} = \lim_{x\to 0} \frac{\cancel{x^{18}}}{8\cancel{x^{18}}} = \frac{1}{8}.$$

2) Consider $y = 2x^3$

$$\lim_{x\to 0} \frac{x^{12}\cdot 4x^6}{(x^6+4x^6)^3} = \lim_{x\to 0} \frac{4x^{18}}{(5x^6)^3} = \lim_{x\to 0} \frac{4\cancel{x^{18}}}{125\cancel{x^{18}}} = \frac{4}{125} \neq \frac{1}{8}$$

so limit DNE.

7. Let $f(x, y) = e^{\cos(2x^2+y^2)+2x+y}$.

   (a) Find the linearization of f at $(0, 0)$.

   (b) Approximate $f(-0.3, 0.3)$ using the linearizatinon in the first problem.

a) $f_x = e^{\cos(2x^2+y^2)+2x+y} \cdot (-\sin(2x^2+y^2) \cdot 4x + 2)$

$f_x(0,0) = e^{\cos(0+0)+0+0} \cdot (-\sin(0) \cdot 0 + 2) = 2e.$

$f_y = e^{\cos(2x^2+y^2)+2x+y} (-\sin(2x^2+y) \cdot 2y + 1)$

$f_y(0,0) = e$

$f(0,0) = e^{\cos(0+0)+0+0} = e.$

So $L(x,y) = e + 2e(x-0) + e(y-0) = e + 2ex + ey.$

b) $L\left(-\frac{3}{10}, \frac{3}{10}\right) = e + 2e\left(\frac{-3}{10}\right) + e\left(\frac{3}{10}\right) = \frac{7e}{10}$

8. Let
$$f(x,y)=\begin{cases}\frac{y^3}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

   (a) Show that $f(x,y)$ is continuous on $\mathbb{R}^2$.

   (b) Prove that $D_u f(0,0)$ exist for all unit vectors $u\in\mathbb{R}^2$.

   (c) Is $f(x,y)$ differentiable at $(0,0)$?

a) Only need to show $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$.

$$\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\sin^3\theta}{r^2} = \lim_{r\to 0} r\sin^3\theta = 0.$$

So f is cts on $\mathbb{R}^2$.

b) Let $u=(a,b)$ with $a^2+b^2=1$. Then

$$D_u f(0,0) = \lim_{h\to 0} \frac{f((0,0)+hu) - f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{f(ah,bh)}{h} = \lim_{h\to 0} \frac{b^3 h^{\cancel{3}}}{\cancel{h^2}(\underbrace{a^2+b^2}_{=1})} = \lim_{h\to 0} b^3 h = 0.$$

c) $$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{y^3}{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \lim_{r\to 0} \frac{\cancel{r^3}\sin^3\theta}{\cancel{r^3}} = \lim_{r\to 0} \sin^3\theta \text{ DNE.}$$

So f is not diff. at $(0,0)$.